Joint PDF of Continuous Random Variables

1. Two Continuous R.V.

Let X and Y be cont. r.v. Then $f(x, y)$ is their joint probability density function or joint PDF for X and Y if for any two-dimensional set A,

$$P[(X, Y) \in A] = \int \int_A f(x, y) dx dy$$

In particular, if A is the 2-D rectangle with $\{a \le X \le b, c \le Y \le d\}$, then we can bound the integral as $\int_c^d \int_a^b f(x, y) dx dy$. Conditions for a joint PDF: (1) it must be non-negative, i.e. $f(x, y) \ge 0$ for all x and y and (2) $\int \int f(x, y) dx dy = 1$.

2. Obtaining Marginal PDF's from Joint PDF's

Given the joint PDF $f(x, y)$ of two cont. r.v., the marginal PDF, or the marginal density, of X and Y, can be obtained by integrating the joint PDF over the other variable; recall the marginal PMF's of discrete r.v. are obtained by summing the joint PMF over values of the other variable.

Joint CDF (for discrete and cont. r.v.)

The joint cumulative distribution function of the k r.v. $X_1, X_2, \ldots, X_k$ is the function defined by

$$F(x_1, \ldots, x_k) = P(X_1 \le x_1, \ldots, X_k \le x_k).$$

The random variables can be discrete or cont., or some be discrete and other be continuous.

3. Joint PDF & CDF for Cont. R.V.

If $f(x, y)$ is the joint PDF for X and Y, their joint CDF is

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du dv.$$

Conversely, if $F(x, y)$ is the joint CDF for X and Y, their joint PDF is

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

4. Independent Random Variables

Recall that two events A and B are independent if $P(A \cap B) = P(A)P(B)$ and similarly two random variables X and Y are independent if $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B.

Two discrete random variables X and Y are independent iff

$$p(x, y) = p_X(x)p_Y(y)$$

for all x and y, i.e. the joint PMF is the product of their marginal PMF's.

Two continuous random variables X and Y are independent iff

$$F(x, y) = F_X(x)F_Y(y)$$

for all x and y, i.e. the joint CDF is the product of their marginal CDF's.

Their joint PDF is

$$f(x, y) = \frac{\partial^2}{\partial x \partial y}F(x, y) = \frac{\partial^2}{\partial x \partial y}F_X(x)F_Y(y) = F_X'(x)F_Y'(y) = f_X(x)f_Y(y)$$

Conversely, if $f(x, y) = f_X(x)f_Y(y)$, their joint PDF is

$$\begin{aligned} F(x, y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f(u, v)\,du\,dv = \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(u)f_Y(v)\,du\,dv \\ &= \int_{-\infty}^{x} f_X(u)\,du \int_{-\infty}^{y} f_Y(v)\,dv = F_X(x)F_Y(y) \end{aligned}$$

i.e. independent iff joint PDF = the product of marginal PDF's.

5. A Simple Method

It seems like we have to find marginal distributions (a lot of work) to check independence. However, there's a simple criterion: X and Y are independent if the joint PMF/PDF can be written as the product of a function of x and a function of y (which ought to be fairly intuitive), i.e.

$$f(x, y) = g(x)h(y), \text{ for all } x,\ y.$$

Here $g(x) \geq 0$ and $h(y) \geq 0$ are not necessarily PMFs/PDFs.